

# CASIMIR ENERGY OF A DILUTE DIELECTRIC BALL AT ZERO AND FINITE TEMPERATURE

V.V. NESTERENKO

*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research  
Dubna, 141980, Russia*

G. LAMBIASE and G. SCARPETTA

*Dipartimento di Scienze Fisiche E.R. Caianiello, Università di Salerno  
Baronissi (SA), 84081, Italy  
INFN, Sezione di Napoli, Napoli, 80126, Italy*

The basic results in calculations of the thermodynamic functions of electromagnetic field in the background of a dilute dielectric ball at zero and finite temperature are presented. Summation over the angular momentum values is accomplished in a closed form by making use of the addition theorem for the relevant Bessel functions. The behavior of the thermodynamic characteristics in the low and high temperature limits is investigated. The  $T^3$ -term in the low temperature expansion of the free energy is recovered (this term has been lost in our previous calculations).

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## 1. Introduction

Calculation of the Casimir energy of a dielectric ball has a rather long history starting 20 years ago.<sup>1</sup> However only recently the final result was obtained for a dilute dielectric ball at zero<sup>2,3</sup> and finite<sup>4,5</sup> temperature. Here we summarize briefly the derivation of the Casimir energy of a dilute dielectric ball by making use of the mode summation method and the addition theorem for the Bessel functions instead of the uniform asymptotic expansion for these functions.<sup>3,4</sup>

## 2. The Basic Steps of Calculations and the Results

A solid ball of radius  $a$  placed in an unbounded uniform medium is considered. The contour integration technique<sup>3</sup> gives ultimately the following representation for the Casimir energy of the ball

$$E = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{y_0} dy y \frac{d}{dy} \ln \left[ W_l^2(n_1 y, n_2 y) - \frac{\Delta n^2}{4} P_l^2(n_1 y, n_2 y) \right], \quad (1)$$

where

$$W_l(n_1 y, n_2 y) = s_l(n_1 y) e_l'(n_2 y) - s_l'(n_1 y) e_l(n_2 y),$$

$$P_l(n_1 y, n_2 y) = s_l(n_1 y) e'_l(n_2 y) + s'_l(n_1 y) e_l(n_2 y),$$

and  $s_l(x)$ ,  $e_l(x)$  are the modified Riccati-Bessel functions,  $n_1, n_2$  are the refractive indices of the ball and of its surroundings,  $\Delta n = n_1 - n_2$ .

Analysis of divergences carried out in our paper<sup>3</sup> leads to the following algorithm for calculating the vacuum energy (1) in the  $\Delta n^2$ -approximation. First, the  $\Delta n^2$ -contribution should be found, which is given by the sum  $\sum_l W_l^2$ . Upon changing its sign to the opposite one, we obtain the contribution generated by  $W_l^2$ , when this function is in the argument of the logarithm. The  $P_l^2$ -contribution into the vacuum energy is taken into account by expansion of Eq. (1) in terms of  $\Delta n^2$ .

Applying the addition theorem for the Bessel functions<sup>6</sup>

$$\sum_{l=0}^{\infty} (2l+1) [s'_l(\lambda r) e_l(\lambda \rho)]^2 = \frac{1}{2r\rho} \int_{r-\rho}^{r+\rho} \left( \frac{1}{\lambda} \frac{\partial \mathcal{D}}{\partial r} \right)^2 R dR$$

with

$$\mathcal{D} = \frac{\lambda r \rho}{R} e^{-\lambda R}, \quad R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}$$

one arrives at the result

$$E = \frac{23}{384} \frac{\Delta n^2}{\pi a} = \frac{23}{1536} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\pi a}, \quad \varepsilon_i = n_i^2, \quad i = 1, 2.$$

Extension to finite temperature  $T$  is accomplished by substituting the  $y$ -integration in (1) by summation over the Matsubara frequencies  $\omega_n = 2\pi n T$ . When considering the low temperature behavior of the thermodynamic functions of a dielectric ball the term proportional to  $T^3$  in our paper<sup>4</sup> was lost. It was due to the following. We have introduced the summation over the Matsubara frequencies in Eq. (3.20) under the sign of the  $R$ -integral. Here we show how to do this summation in a correct way.

In the  $\Delta^2$ -approximation the last term in Eq. (3.20) from the article<sup>4</sup>

$$\overline{U}_W(T) = 2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 \int_{\Delta n}^2 \frac{e^{-2w_n R}}{R} dR, \quad w_n = 2\pi n a T \quad (2)$$

can be represented in the following form

$$\overline{U}_W(T) = -2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 E_1(4w_n), \quad (3)$$

where  $E_1(x)$  is the exponential-integral function.<sup>7</sup> Now we accomplish the summation over the Matsubara frequencies by making use of the Abel-Plana formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x) dx + i \int_0^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} dx. \quad (4)$$

The first term in the right-hand side of this equation gives the contribution independent of the temperature, and the net temperature dependence is produced by the second term in this formula. Being interested in the low temperature behavior of the internal energy we substitute into the second term in Eq. (4) the following expansion of the function  $E_1(z)$

$$E_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k \cdot k!}, \quad |\arg z| < \pi, \quad (5)$$

where  $\gamma$  is the Euler constant.<sup>7</sup> The contribution proportional to  $T^3$  is produced by the logarithmic term in the expansion (5). The higher powers of  $T$  are generated by the respective terms in the sum over  $k$  in this formula ( $t = 2\pi aT$ )

$$\overline{U}_W(T) = \frac{\Delta n^2}{\pi a} \left( -\frac{1}{96} + \frac{\zeta(3)}{4\pi^2} t^3 - \frac{1}{30} t^4 + \frac{8}{567} t^6 - \frac{8}{1125} t^8 + \mathcal{O}(t^{10}) \right). \quad (6)$$

All these terms, save for  $2\zeta(3)\Delta n^2 a^2 T^3$ , are also reproduced by the last term in Eq. (3.31) in our paper<sup>4</sup> (unfortunately additional factor 4 was missed there)

$$\frac{\Delta n^2}{8} T \cdot 4 t^2 \int_{\Delta n}^2 \frac{dR}{R} \frac{\coth(tR)}{\sinh^2(tR)}.$$

Taking all this into account we arrive at the following low temperature behavior of the internal Casimir energy of a dilute dielectric ball

$$U(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} + \frac{\zeta(3)}{4\pi^2} t^3 - \frac{7}{360} t^4 + \frac{22}{2835} t^6 - \frac{46}{7875} t^8 + \mathcal{O}(t^{10}) \right). \quad (7)$$

The relevant thermodynamic relations give the following low temperature expansions for free energy

$$F(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} - \frac{\zeta(3)}{8\pi^2} t^3 + \frac{7}{1080} t^4 - \frac{22}{14175} t^6 + \frac{46}{55125} t^8 + \mathcal{O}(t^{10}) \right) \quad (8)$$

and for entropy

$$S(T) = -\frac{\partial F}{\partial T} = \Delta n^2 \left( \frac{3\zeta(3)}{4\pi^2} t^2 - \frac{7}{135} t^3 + \frac{88}{4725} t^5 - \frac{736}{55125} t^7 + \mathcal{O}(t^9) \right). \quad (9)$$

The range of applicability of the expansions (7), (8), and (9) can be roughly estimated in the following way. The curve  $S(T)$  defined by Eq. (9) monotonically goes up when the dimensionless temperature  $t = 2\pi aT$  changes from 0 to  $t \sim 1.0$ . After that this curve sharply goes down to the negative values of  $S$ . It implies that Eqs. (7) – (9) can be used in the region  $0 \leq t < 1.0$ . The  $T^3$ -term in Eqs. (7) and (8) proves to be principal because it gives the first positive term in the low temperature expansion for the entropy (9). It is worth noting, that the exactly the same  $T^3$ -term, but with opposite sign, arises in the high temperature asymptotics of free energy in the problem at hand (see Eq. (4.30) in Ref.<sup>8</sup>).

For large temperature  $T$  we found<sup>4</sup>

$$U(T) \simeq \frac{\Delta n^2}{8} T, \quad F(T) \simeq -\frac{\Delta n^2}{8} T [\ln(aT) - c], \quad S(T) \simeq \frac{\Delta n^2}{8} [\ln(aT) + c + 1], \quad (10)$$

where  $c$  is a constant<sup>5,8</sup>  $c = \ln 4 + \gamma - 7/8$ . Analysis of Eqs. (3.20) and (3.31) from the paper<sup>4</sup> shows that there are only exponentially suppressed corrections to the leading terms (10).

The Casimir forces, exerted on the surface of a dielectric ball and tending to expand it, have the following low temperature and high temperature asymptotics

$$\mathcal{F} = -\frac{1}{4\pi a^2} \frac{\partial F(T)}{\partial a} = \frac{23}{1536} \frac{\Delta n^2}{\pi^2 a^4} \left( 1 + \frac{96}{23} \frac{\zeta(3)}{\pi^2} t^3 - \frac{112}{345} t^4 + \mathcal{O}(t^6) \right),$$

$$\mathcal{F} \simeq \frac{\Delta n^2}{32\pi a^3} T, \quad T \rightarrow \infty.$$

Summarizing we conclude that now there is a complete agreement between the results of calculation of the Casimir thermodynamic functions for a dilute dielectric ball carried out in the framework of two different approaches: by the mode summation method<sup>3,4</sup> and by perturbation theory for quantized electromagnetic field, when dielectric ball is considered as a perturbation in unbounded continuous surroundings.<sup>5</sup>

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